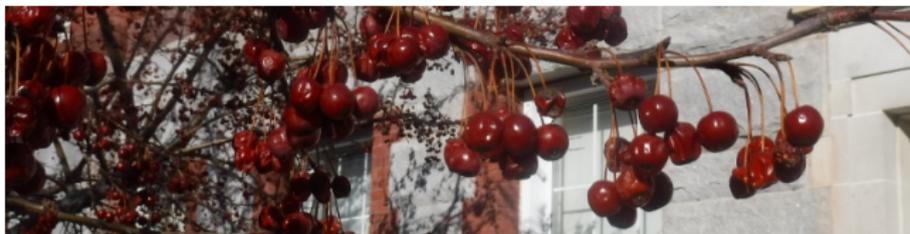


The random graph

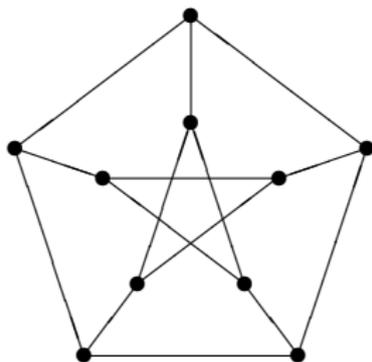
Peter J. Cameron
University of St Andrews

Mathematics Colloquium
Worcester Polytechnic Institute
29 March 2019



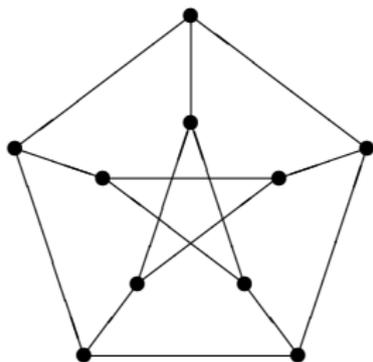
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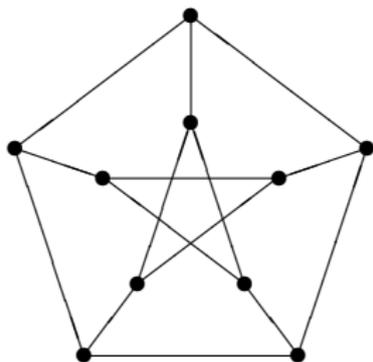
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I am going to tell you about the most famous *infinite* graph ...

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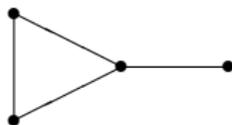
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Graphs and induced subgraphs

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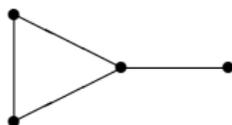
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Rado's universal graph



In 1964, Richard Rado published a construction of a countable graph which was **universal**. This means that every finite or countable graph occurs as an induced subgraph of Rado's graph.

Rado's construction

The vertex set of Rado's graph R is the set \mathbb{N} of natural numbers (including 0).

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Theorem

There is a countable graph R with the following property: if a random graph X on a fixed countable vertex set is chosen by selecting edges independently at random with probability $\frac{1}{2}$, then the probability that X is isomorphic to R is equal to 1.

The proof

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I claim that one of the distinguishing features of mathematics is that you can be convinced of such an outrageous claim by some simple reasoning. I do not believe this could happen in any other subject.

Property (*)

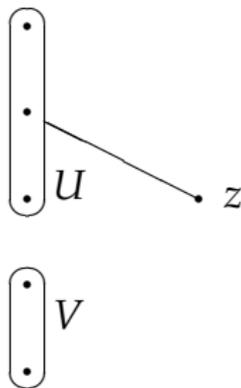
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The point z is called a **witness** for the sets U and V .

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Then you will be convinced!

Proof of Fact 1

We use from measure theory the fact that a countable union of null sets is null. We are trying to show that a countable graph fails (*) with probability 0; since there are only countably many choices for the (finite disjoint) sets U and V , it suffices to show that for a fixed choice of U and V the probability that no witness z exists is 0.

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So the event that no witness exists has probability 0, as required.

Proof of Fact 2

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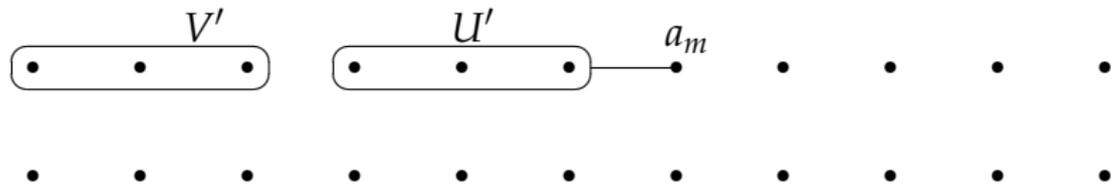
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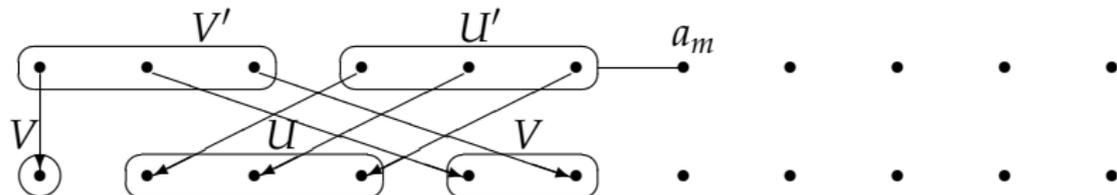


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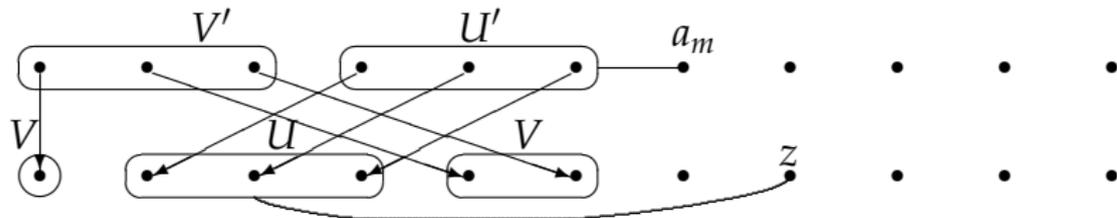


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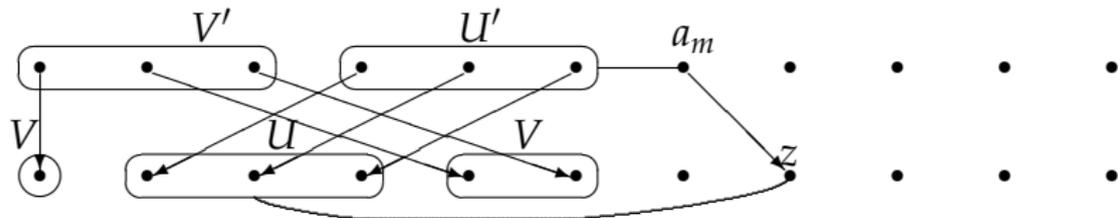


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The back-and-forth method is often credited to Georg Cantor, but it seems that he never used it, and it was invented later by E. V. Huntington.

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I mentioned the problem of finding a non-trivial symmetry of this graph. There seems to be no simple formula for one! But we know it must exist. Indeed, there is a **primitive recursive** automorphism.

A number-theoretic construction

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So we can construct a graph whose vertices are all the prime numbers congruent to 1 (mod 4), with p and q joined if and only if p is a quadratic residue (mod q): the law of quadratic reciprocity guarantees that the edges are undirected.

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By the Chinese Remainder Theorem, the simultaneous congruences

- ▶ $z \equiv a_i \pmod{u_i}$ for all $u_i \in U$,
- ▶ $z \equiv b_j \pmod{v_j}$ for all $v_j \in V$,
- ▶ $z \equiv 1 \pmod{4}$,

have a solution modulo $4 \prod u_i \prod v_j$. By Dirichlet's Theorem, this congruence class contains a prime, which is the required witness.

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My point here is not to resolve this paradox, but to use it constructively.

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Indeed, the precise form of the axioms is not so important. We need a few basic axioms (Empty Set, Pairing, Union) and, crucially, the Axiom of Foundation, and that is all. It does not matter, for example, whether or not the Axiom of Choice holds.

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When we apply the construction of "symmetrising the membership relation" to this model, we obtain Rado's description of his graph!

Group-theoretic properties

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- ▶ $\text{Aut}(R)$ has cardinality 2^{\aleph_0} .
- ▶ $\text{Aut}(R)$ is simple.
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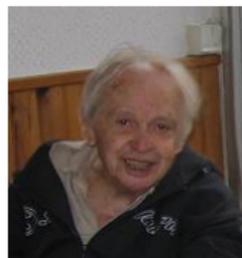
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- ▶ R is a **Cayley graph** for a wide class of countable groups, including all countable abelian groups of infinite exponent. For these groups, a “random Cayley graph” is isomorphic to R with probability 1.

Rolling back the years, 1



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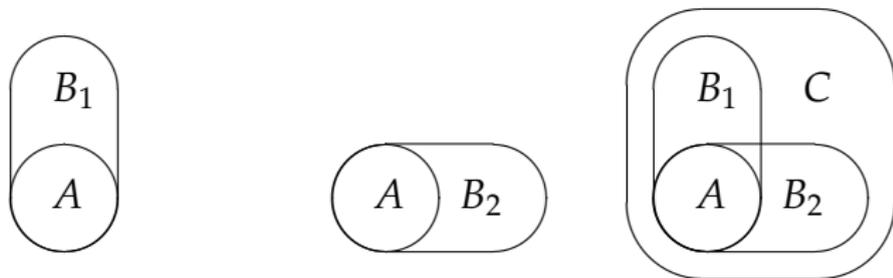
A class \mathcal{C} satisfying these conditions is a **Fraïssé class**, and the countable homogeneous structure M is its **Fraïssé limit**.

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The amalgamation property says that two structures B_1, B_2 in the class \mathcal{C} which have substructures isomorphic to A can be “glued together” along A inside a structure $C \in \mathcal{C}$:

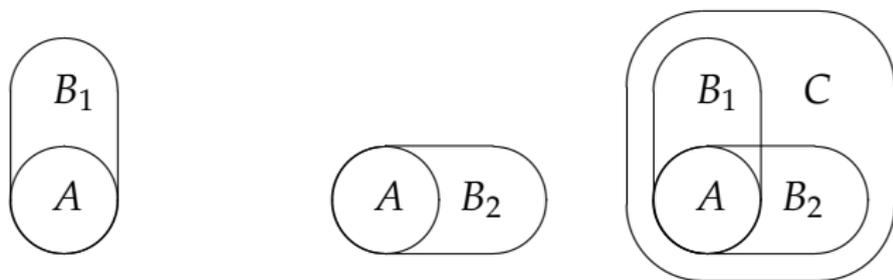
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Note that the intersection of B_1 and B_2 may be larger than A .

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Each of the following classes is a Fraïssé class; the proofs are exercises. Thus the corresponding universal homogeneous Fraïssé limits exist.

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There are *many* others!

Rolling back further



A quarter of a century earlier, these ideas had already been used by the Soviet mathematician P. S. Urysohn. He visited western Europe with Aleksandrov, talked to Hilbert, Hausdorff and Brouwer, and was drowned while swimming in the sea at Batz-sur-Mer in south-west France at the age of 26 in 1924.

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Urysohn's theorem

A **Polish space** is a metric space which is **complete** (Cauchy sequences converge) and **separable** (there is a countable dense set). A metric space M is **homogeneous** if any isometry between finite subspaces extends to an isometry of M .

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This unique metric space is known as the **Urysohn space**. Its study has been popularised in recent years by Anatoly Vershik.

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Its completion is easily seen to be the required Polish space.

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Let M be the Fraïssé limit of the class of metric spaces with all distances 1 or 2; form a graph by joining two points if their distance is 1. Since the graph is homogeneous, if v and w are two vertices at distance 2, there is a vertex at distance 1 from both. Thus the distance in M coincides with the graph distance in this graph. The graph is universal and homogeneous, and so is R .